CODES OVER RINGS

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PURE PROGRAM FOR UNDERGRADUATE RESEARCH

ABSTRACT

In order to investigate linear codes, it is important to factorize polynomials of the type $x^m - \lambda$ over various chain rings into its basic irreducible factors and determine which of these factors are self-reciprocal. In this project we investigate this problem over more general rings when m is a odd prime number.

OBJECTIVES

The initial objective was to work on the following conjecture [1].

Conjecture 3.5 Assume that m is an odd prime and $\gcd(m, q) = 1$, where q is a prime power. Let $\alpha \mid (m-1)$ and $\mathrm{ord}_m(q) = \frac{m-1}{\alpha}$, we can cast the factorization of $x^m - \lambda$ into distinct basic irreducible polynomials over $R_k = \frac{\mathbb{F}_q[u]}{\langle u^k \rangle}$ as follows.

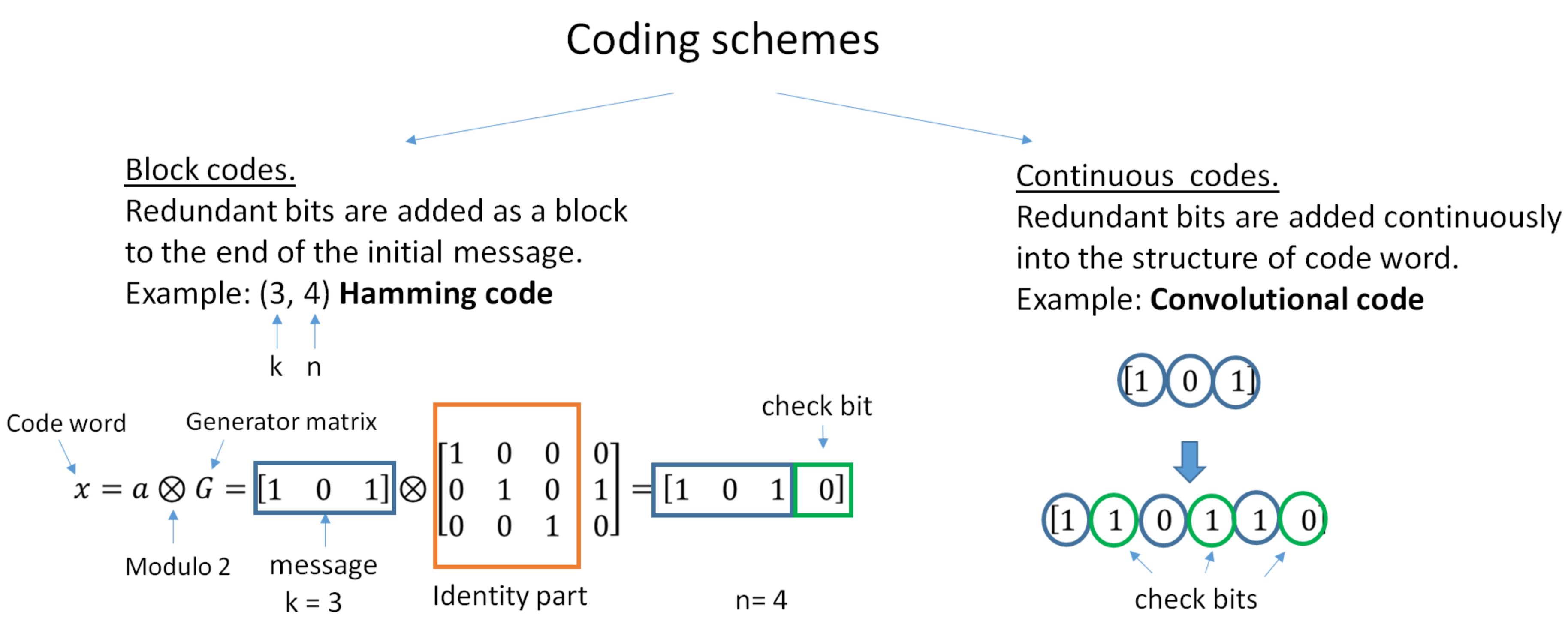
(1) If α is an odd integer, then we have $x^m - \lambda = A(x)\prod_{i=1}^{\alpha} g_i(x)$, where $g_i(x) = g_i^*(x)$, $\deg(g_i(x)) = \frac{m-1}{\alpha}$;

(2) If α is an even integer, then we have $x^m - \lambda = A(x)\prod_{j=1}^{\frac{\alpha}{2}} h_j(x)h_j^*(x)$, where $\deg(h_j(x)) = \deg(h_j^*(x)) = \frac{m-1}{\alpha}$; if

(i) $\lambda = 1$, $A(x) = x - 1$, or
(ii) $\lambda = -1$, $A(x) = x + 1$, or
(iii) $\lambda = 1 + \omega u^t$, q is a power of 2, $A(x) = x + 1 + \omega u^t$, where $t \geq \lceil \frac{k}{2} \rceil$, $\omega \in R_k^\times$.

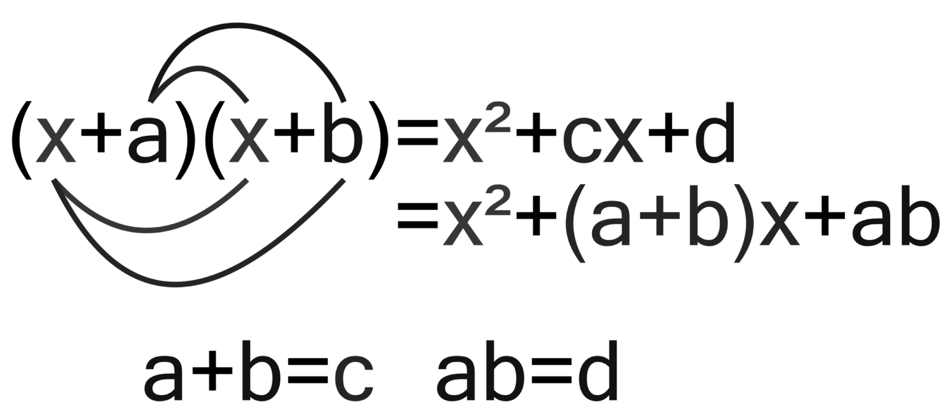
PROJECT DETAILS

- A list of messages can be modelled as vector subspaces over "finite fields" and such subspaces are called "code".
- Modelling a code as a vector space allows us to do linear algebra to it and this makes predicting and analyzing the structure of the codes more convenient.
- We can also use certain algebraic tools and methods of linear algebra to construct a basis for a linear code.



[4]

- Traditional codes are vector subspaces with coefficients from a finite field $\mathbb{F}_q$, such as $\mathbb{Z}_p$ for a prime p.
- One of the benefits of using such algebraic structures is the unique factorization property.



[3]

- Cyclic linear codes can be seen as ideals in $\mathbb{F}_q[X] / <x^m-1>$. Every ideal is principle and generators of these ideals are exactly the factors of x^m-1 over this ring.
- We want to factorize $x^m - \lambda$ over $\mathbb{F}_q[u] / <u^k>$.
- Let $x^m - \lambda = f_1(x)f_2(x)\ldots f_k(x)$ over $\mathbb{F}_q[u] / <u^k>$ where each $f_i(x)$ is irreducible . Then there are 2^k cyclic codes of that length m.
- By the Chinese Reminder Theorem;

$$\frac{R[X]}{<x^m - \lambda>} \cong \frac{R[X]}{<f_1>} \oplus \ldots \oplus \frac{R[X]}{<f_k>}$$

where $R = \mathbb{F}_q[u] / <u^k>$ and $R[X]/<f_i>$ is a field for all i.

- This corresponds to the decomposition of linear codes over $R[X]/<x^m - \lambda>$;

$$C \cong C_1 \oplus C_2 \oplus \ldots \oplus C_k$$

- As a method to factorize a polynomial $f(x)$ over $\mathbb{F}_q[u] / <u^k>$ we can quotient through the ideal $<u>$, use the unique factorization in $\mathbb{F}_q$ and lift the factorization to $\mathbb{F}_q[u] / <u^k>$ via Hensel's lemma [2].
- In fact, we can generalize this factorization to any finite local commutative ring.

CONCLUSION

Over a finite local commutative ring R with the residue field K, let π be the natural quotient map and $char(K) = p$. Define $\ell = \mathrm{ord}_m(q)$, $\alpha = (m-1)/\ell$. Let f be a polynomial in $R[X]$ with $\pi(f) = x^m \pm 1$ in $K[X]$.

We conclude that the conjecture is wrong when both α and ℓ are even. We also proved the following cases;

- When α is odd;
  f splits into a linear component $A(x)$ with $\pi(A(x)) = x \pm 1$ and α many self-reciprocal basic irreducible polynomials of degree ℓ.
- When α is even and ℓ is odd;
  f splits into a linear component $A(x)$ with $\pi(A(x)) = x \pm 1$ and $\alpha/2$ many pair of reciprocal basic irreducible polynomials of degree ℓ.

REFERENCES

[1]Qian, Liqin et al. "On self-dual and LCD quasi-twisted codes of index two over a special chain ring." *Cryptography and Communications* (2018)

[2] McDonald, Bernard R. (1974). *Finite rings with identity*. New York : M. Dekker

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