KANADE RUSSELL'S IDENTITY FINDER

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Abstract

In this project, our first aim is to get familiar with the integer partitions which are restricted by conditioning. We proposed a framework which is based on finding a new identity that has not been discovered before. We studied the integer partitions from the book of Integer Partitions. Comprehending identities and their generating functions enabled us to code those identities in Maxima. These were the fundamental identities of our subject. We found restricted partitions from n=1 to n=20 by using Maxima codes. We examined the paper of Kanade Russell's IdentityFinder and coded the generalization of partitions. This allowed us to find a's from the b's of identities. Moreover, each identity in different papers is examined and implemented the code. In the end, we searched for a new identity by restoring conditions.

Keywords: Integer partitions, Maxima, Identity, Roger-Ramanujan, Generating functions

1 Introduction

Integer partition is a way of writing an integer to sum of positive integers. For example, the partitions of 4 are: [4], [3,1], [2,2], [2,1,1], [1,1,1,1]. The partition summands are known as parts. Number of n partitions is provided by the partition function p(n). For instance, the number of partitions of 4 is 5 and can be shown as p(4)=5.

n	1	2	3	4	6	7	8	9	10
p(n)	1	2	3	5	7	11	15	22	42

Rather than just p(n), some conditions can be implemented to the partitions such as the partitions can only be odd numbers. These conditions can be shown as p(n|condition A). If there are two conditions such that **p(n|condition A) = p(n|condition B)**, this is called a partition identity.

Euler was a famous 18th century mathematician who works in different issues of mathematics such as geometry, calculus, number theory, lunar theory and integer partitions. Euler states that “Every number has as many integer partitions into odd parts as into distinct parts.”[1].

p(n | odd parts) = p(n | distinct parts) [2]

For n = 4, distinct parts of n are [3,1] and [4]. This means that the number of distinct parts of 4 is two. According the Euler’s identity there should be two odd parts of 4 and these are [3,1] and [1,1,1,1]. The identity is valid for all values of n.

Roger-Ramanujan were famous mathematicians who made major contributions in the field of Integer Partitions. They found an expression that helps describing some sort of partition identities in general meaning.

p(n | [some condition]) = p(n | parts in N) for all n > 0 [3]

They also found two well known identities that fit the expression that they made. First identity that they found implies that number of partitions of any number using only the set of numbers which consist of a set of numbers that are congruent to 1 or 4 in mod 5 is equal to the number of partitions of the same number by using the numbers that have at least 2 differences between each other.

p(n | parts ≡ 1 or 4 (mod 5)) = p(n | 2-distinct parts) [4]

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[1] Andrews and Eriksson, 2004: 3
[2] Andrews and Eriksson, 2004: 3
[3] Andrews and Eriksson, 2004: 29
[4] Andrews and Eriksson, 2004: 31

Second identity that they found was similar to the first one. Left side of the bijection is parts by using only the elements of the set of numbers that congruent to 2 or 3 in mod 5 and right side is 2-distinct parts and all parts are bigger than 1.

p(n | parts ≡ 2 or 3 (mod 5)) = p(n | 2-distinct parts, each part > 1) [5]

Generating Functions are expressions that explain infinite power series and also usually being used for tracking number sequences by looking at the coefficients or powers of elements which are symbolized with "q". They also can be used for expressing the Integer Partition Identities. For example,

$$\sum_{n=0}^{\infty} p(n \mid \text{parts distinct}) q^n = \prod_{n=1}^{\infty} (1+q^n)$$ [6]

$$\sum_{n=0}^{\infty} p(n \mid \text{parts all odd}) q^n = \prod_{n \text{ odd}} \frac{1}{(1-q^n)}$$ [7]

Maxima is a computer algebra system. Instead of numeric calculation the program can do algebra. In this way, the operations in the Maxima is much faster than other numeric programs. The Maxima has also a unique programming language. Thus, the Maxima allows to find integer partitions and number of integer partitions of a given condition.

IdentityFinder is an algorithm found by Shashank Kanade, Matthew C. Russell and written in the article called IdentityFinder and Some New Identities of Rogers–Ramanujan Type . The algorithm can be programmed in different languages and basically helps users to determine whether a given set of numbers belong to a type of partition's counts or not.

$$nb_n = na_n + \sum_{d \mid n,\, d<n} da_d + \sum_{j=1}^{n-1} \left(\sum_{d \mid j} da_d \right) b_{n-j}$$ [8]

This project is about finding new identities by using IdentityFinder algorithm.

2 Kanade and Russel's IdentityFinder

Project had two main steps. First, acquiring background knowledge and experience to understand, code and construct identities. Second, using the background knowledge and coding experience in Maxima to implement Kanade Russel's IdentityFinder algorithm and discovering or rediscovering genuine identities.

[5] Andrews and Eriksson, 2004: 33
[6] Andrews and Eriksson, 2004: 47
[7] Andrews and Eriksson, 2004: 47
[8] Kanade and Russell, 2015: 420

2.1 Examination and Verification of Some Identities

At the first stage of project, integer partitioning methods and different identities such that mentioned in the introduction are examined. Integer Partitions by George E.Andrews and Kimmo Eriksson was the prime source while inspecting the identities and some simple bijective proofs of basic identities such as Euler's Identity[9].And after that more than 30 identities mentioned in the book such as Roger-Ramajuan 1, Roger-Ramajuan 2 are verified until 20 or more in some cases to generate a basis for Kanade Russel's IdentityFinder algorithm[10]. The main purpose of examination and verification of some identities is to have enough background knowledge of different fundamental identities to be able to construct genuine identities .

Proof of identities require more complex and a wider background that's why such proofs are not part of our project so many proofs in the book are overlooked though a few simple bijective proofs are important to understand why looking for identities are required rather than counting methods for each integer partition. One important bijection proof is for Euler's identity which constructs a bijection between p(n|distinct parts) and p(n|odd parts) through split and merge processes[11]. A partition with odd numbers can be transformed into a unique partition with distinct parts by merging any equal pair until there are no more equal pairs. And in reverse splitting any part to halves until there is no even part[12]. Examining bijection was an important part to understand the pursuit of new identities.

Verifying some fundamental identities were the second stage of the project which was important for two reasons. One for practicing to code various restrictions of partitioning such as distinct, odd, 2-distinct (distance between consecutive elements are at least 2), number of even parts are odd and many more. In addition to that creating a basis for Kanade Russell's IdentityFinder algorithm was another important purpose. So that a sample of results was ready to test the algorithm.

```
print("Number of Partitions into even number of Only Odd Parts")$
print(" --- ")$
checkSetLengthEven(x) := if evenp(length(x)) then block(return(true)) else block(return(false))$
for i from 1 thru 20 do(
    print("For: ", i),
    s:integer_partitions(i),
    subS: subset (s, lambda ([x], every (oddp, x))),
    /·disp(subS)·/
    subS2: subset(subS, lambda([x], checkSetLengthEven(x))),
    /·disp(subS2),·/
    print("Number of partitions: ", length(subS2))
);
```

[9] Andrews and Eriksson, 2004: 5
[10] Kanade and Russell, 2015: 420
[11] Andrews and Eriksson, 2004: 8
[12] Andrews and Eriksson, 2004: 9

Figure 2.1.1

```
print("Number of Partitions such that n = x1 + x2 where x2 >= x1 >= 0")$
print(" --- ")$

for i from 1 thru 20 do(
   s: integer_partitions(i),
   a:subset(s, lambda([x], if length(x)=2  then block( return(true)) else block(return(false)))),
   check1: 1,
   print("For: ", i),
   countPart: 1, /· Each partition includes itself ·/
   for prt in a do(
      /·
      disp(prt[1]),
      disp(prt[2]),·/
      if prt[1] >=  (2·prt[2]) or (2·prt[1]) <= prt[2] then check1: 0,
      if check1 = 0 then disp(prt) and countPart: countPart + 1
   ),

   print("Number of partitions: ", countPart)
);
```

Figure 2.1.2

As shown in Figures 2.1.1 and 2.1.2 number of partitions are found for a given n which is 20 for many cases because it was enough to see the pattern of powers of elements which emerged in the algorithm.

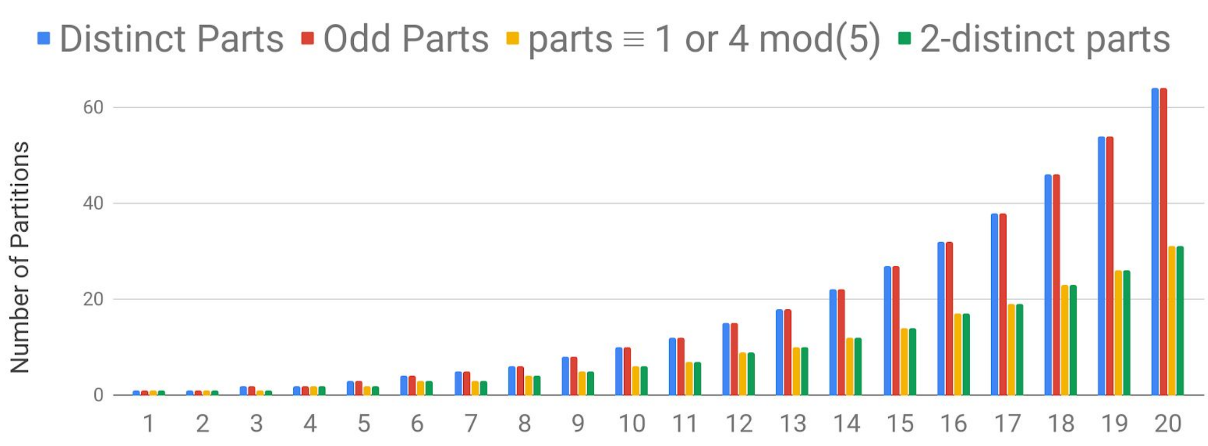

Chart 2.1

Verification for two identities which are Euler's Identity and Roger-Ramajuan 1, are clear in chart 2.1 for n greater than zero until twenty. Identities are also checked from https://oeis.org/ to have reliable test cases for the algorithm.

2.2 Encoding Euler's Algorithm and Examination of More Identities

The main part of the project started after reading the paper called Kanade-Russell's IdentityFinder. As well as some identities, the paper describes an algorithm which allows us to find the first N terms of the sequence of the {a□} product side[13].

Proposition 1. (Cf. Theorem 10.3 of [Andrews 86].) Let $f(q)$ be a formal power series such that

$$f(q) = 1 + \sum_{n \geq 1} b_n q^n. \tag{2–1}$$

Then

$$f(q) = \prod_{m \geq 1} (1 - q^m)^{-a_m}, \tag{2–2}$$

where the a_ms are defined recursively by

$$nb_n = na_n + \sum_{d|n,\, d<n} da_d + \sum_{j=1}^{n-1} \left(\sum_{d|j} da_d \right) b_{n-j}. \tag{2–3}$$

Figure 2.2.1: Euler's algorithm used in IdentityFinder

(2-1):"sum side" (2-2):"product side" (2-3): "Equation for "factoring" a sum side into infinite product"

Next step was to encode the algorithm in figure 2.2.1, using Maxima. However as a group, we decided to encode the algorithm using a language we are more familiar with first, which is C++, to make things a little easier for us. After managing to do that, with the request of our supervisor, we transposed the code to Maxima.

[13] Kanade and Russell, 2015: 420

```
a: makelist(0,num_of_partitions)$
for n: 1 thru length(b)
do(
   part1:0, d:1,
   for d thru n−1
      do (
         if(mod(n,d)=0) then (part1: part1+d·a[d])
         ),
    part2: 0,
    j:1,

   for j thru n−1 do
      (
      part2_inside:0,
      second_d:1,
      for second_d thru j do(
            if(mod(j,second_d)=0)
            then
            ( part2_inside: part2_inside + second_d·a[second_d])
         ),
         part2: part2 + part2_inside·b[n−j]
      ),
      a[n]: (n·b[n] − (part1+part2))/n
);
```

Figure 2.2.2: Euler's algorithm encoded in Maxima

We tested the code in figure 2.2.2, using the partition results, first N terms of the sequence of the {b□} sum side, that we found and transferred to an excel document in the earlier stages of the project. Encoding the algorithm was crucial for our project since in the later stages we were going to try to come up with our own identities and in order to do that we would have to test a lot of conditions. Our supervisor recommended some articles for us to read and encode the identities included in the articles, like we did earlier in the project. We encoded 8 more identities for the range [1, 20], this time also using Euler's algorithm, which are (with results):

Mod 9 identities:[14]

I_1 : The number of partitions of a non-negative integer into parts congruent to ± 1 or ± 3 (mod 9) is the same as the number of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

b = [1,1,2,2,2,4,4,5,7,8,9,13,14,16,21,24,27,35,39,45]

a = [1,0,1,0,0,1,0,1,0,1,0,1,0,0,1,0,1,0,1,0]

[14] Kanade and Russell, 2015: 421

I_2 : The number of partitions of a non-negative integer into parts congruent to ± 2 or $\pm 3 \pmod 9$ is the same as the number of partitions with smallest part at least 2 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

b=[0,1,1,1,1,3,2,3,4,4,5,8,7,10,12,13,15,21,20,26]

a=[0,1,1,0,0,1,1,0,0,0,1,1,0,0,1,1,0,0,0,1]

I_3 : The number of partitions of a non-negative integer into parts congruent to ± 3 or $\pm 4 \pmod 9$ is the same as the number of partitions with smallest part at least 3 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

b=[0,0,1,1,1,2,1,2,3,3,3,6,5,6,9,9,10,15,14,17]

a=[0,0,1,1,1,1,0,0,0,0,0,1,1,1,1,0,0,0,0,0]

I_4 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, or 8 $\pmod 9$ is the same as the number of partitions with smallest part at least 2 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is congruent to 2 $\pmod 3$.

b=[0,1,1,1,2,2,2,4,3,5,6,7,8,11,11,15,17,19,23,28]

a=[0,1,1,0,1,0,0,1,0,0,1,1,0,1,0,0,1,0,0,1]

Mod 12 identities[15]:

I_5 : The number of partitions of a non-negative integer into parts congruent to 1, 3, 4, 6, 7, 10, or 11 (mod 12) is the same as the number of partitions with at most one appearance of the part 1 and difference at least 3 at distance 3 such that if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 1 (mod 3).

b=[1,1,2,3,3,5,7,8,10,14,17,21,27,33,40,50,60,73,89,106]

a=[1,0,1,1,0,1,1,0,0,1,1,0,1,0,1,1,0,1,1,0]

I_6 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, 6, 7, 8, or 11 (mod 12) is the same as the number of partitions with smallest part at least 2, at most one appearance of the part 2, and difference at least 3 at distance 3 such that if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 2 (mod 3).

b=[0,1,1,1,2,3,3,5,5,7,9,11,13,18,20,25,31,37,44,55]

a=[0,1,1,0,1,1,1,1,0,0,1,0,0,1,1,0,1,1,1,1]

The Rogers-Ramanujan Identities and Gordon's Generalization[16]:

THEOREM 7.5. *Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(b_1 b_2 \cdots b_s)$, where $b_j - b_{j+k-1} \geqslant 2$, and at most $i-1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.*

b=[1,2,2,3,4,6,7,10,12,16,19,25,30,38,46,57,68,84,99,121,143,172,202,242,283,336,392,462,537,630]

a=[1,1,0,0,1,1,0,1,1,0,0,1,1,0,1,1,0,0,1,1,0,1,1,0,0,1,1,0,1,1]

[15] Kanade and Russell, 2015: 421
[16] Andrews, 1976: 109

The Gollnitz-Gordon Identities and Their Generalization[17]:

THEOREM 7.11. *Let i and k be integers with $0 < i \leqslant k$. Let $C_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 2 \pmod 4$ and $\not\equiv 0, \pm(2i-1) \pmod{4k}$. Let $D_{k,i}(n)$ denote the number of partitions $(b_1 b_2 \cdots b_s)$ of n in which no odd part is repeated, $b_j \geqslant b_{j+1}$, $b_j - b_{j+k-1} \geqslant 2$ if b_j odd, $b_j - b_{j+k-1} > 2$ if b_j even, and at most $i-1$ parts are $\leqslant 2$. Then $C_{k,i}(n) = D_{k,i}(n)$.*

b=[1,1,2,3,3,4,5,7,9,10,13,17,20,23,29,36,42,49,59,71]

a=[1,0,1,1,0,0,0,1,1,0,1,0,1,0,1,1,0,0,0,1]

The final step was to make some changes on these identities and see whether we can derive new identities.

3 Discussion and Conclusion

As it was mentioned earlier in this report, our main goal was to find new identities from the identities that we studied. We have always been aware of the possibility that we could not find a new identity. Even though, during the project we have never gave up and always worked hard. However, we sadly announce that we could not discover any identity. Integer partition is a relatively new and progressive area that’s why we think that people who will work on this area will have the same motivation. We hope that they can contribute to this area.

[17] Andrews, 1976: 114

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